

# NONUNIFORM DICHOTOMY SPECTRUM AND REDUCIBILITY FOR NONAUTONOMOUS DIFFERENCE EQUATIONS

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**ABSTRACT.** For nonautonomous linear difference equations, we introduce the notion of the so-called nonuniform dichotomy spectrum and prove a spectral theorem. Moreover, we introduce the notion of weak kinematical similarity and prove a reducibility result by the spectral theorem.

## 1. Introduction

Let  $A_k \in \mathbb{R}^{N \times N}$ ,  $k \in \mathbb{Z}$ , be a sequence of invertible matrices. In this paper, we consider the following nonautonomous linear difference equations

$$(1.1) \quad x_{k+1} = A_k x_k,$$

where  $x_k \in \mathbb{R}^N$ ,  $k \in \mathbb{Z}$ . Let  $\Phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ ,  $(k, l) \mapsto \Phi(k, l)$ , denote the evolution operator of (1.1), i.e.,

$$\Phi(k, l) = \begin{cases} A_{k-1} \cdots A_l, & \text{for } k > l, \\ \text{Id}, & \text{for } k = l, \\ A_k^{-1} \cdots A_{l-1}^{-1}, & \text{for } k < l. \end{cases}$$

Obviously,  $\Phi(k, m)\Phi(m, l) = \Phi(k, l)$ ,  $k, m, l \in \mathbb{Z}$ , and  $\Phi(\cdot, l)\xi$  solves the initial value problem (1.1),  $x(l) = \xi$ , for  $l \in \mathbb{Z}$ ,  $\xi \in \mathbb{R}^N$ .

An invariant projector of (1.1) is defined to be a function  $P : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$  of projections  $P_k$ ,  $k \in \mathbb{Z}$ , such that for each  $P_k$  the following property holds

$$P_{k+1}A_k = A_kP_k, \quad k \in \mathbb{Z}.$$

We say that (1.1) admits an *exponential dichotomy* if there exist an invariant projector  $P$  and constants  $0 < \alpha < 1, K \geq 1$  such that

$$(1.2) \quad \|\Phi(k, l)P_l\| \leq K\alpha^{k-l}, \quad k \geq l,$$

and

$$(1.3) \quad \|\Phi(k, l)Q_l\| \leq K(\frac{1}{\alpha})^{k-l}, \quad k \leq l,$$

where  $Q_l = \text{Id} - P_l$  is the complementary projection.

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The notion of exponential dichotomy was introduced by Perron in [28] and has attracted a lot of interest during the last few decades because it plays an important role in the study of hyperbolic dynamical behavior of differential equations and difference equations. For example, see [1, 24, 31] and the references therein. We also refer to the books [17, 21, 25] for details and further references related to exponential dichotomies. On the other hand, during the last decade, inspired both by the classical notion of exponential dichotomy and by the notion of nonuniformly hyperbolic trajectory introduced by Pesin (see [7]), Barreira and Valls have introduced the notion of nonuniform exponential dichotomies and have developed the corresponding theory in a systematic way [8, 9, 10, 11, 12, 13, 14, 15]. As explained by Barreira and Valls, in comparison to the notion of exponential dichotomies, nonuniform exponential dichotomy is a useful and weaker notion. A very general type of nonuniform exponential dichotomy has been considered in [5, 6, 19].

We say that (1.1) admits a *nonuniform exponential dichotomy* if there exist an invariant projector  $P$  and constants  $0 < \alpha < 1, K \geq 1, \varepsilon \geq 1$ , such that

$$(1.4) \quad \|\Phi(k, l)P_l\| \leq K\alpha^{k-l}\varepsilon^l, \quad k \geq l,$$

and

$$(1.5) \quad \|\Phi(k, l)Q_l\| \leq K(\frac{1}{\alpha})^{k-l}\varepsilon^l, \quad k \leq l.$$

When  $\varepsilon = 1$ , (1.4)-(1.5) become (1.2)-(1.3), and therefore a nonuniform exponential dichotomy becomes an exponential dichotomy. For example, given  $\omega > a > 0$ , then the linear equation

$$(1.6) \quad u_{k+1} = e^{-\omega+ak(-1)^k-a(k-1)(-1)^{(k-1)}}u_k, \quad v_{k+1} = e^{\omega-ak(-1)^k+a(k-1)(-1)^{(k-1)}}v_k$$

admits a nonuniform exponential dichotomy, but does not admit an exponential dichotomy. In fact, we have

$$\Phi(k, l)P_l = \begin{pmatrix} e^{-\omega(k-l-1)-a(k-l-1)(-1)^{k-1}-al(-1)^{(k-1)}+al(-1)^l} & 0 \\ 0 & 0 \end{pmatrix}$$

with  $P_l = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Therefore (1.4) holds with

$$K = e^{\omega-a} > 1, \quad \alpha = e^{(-\omega+a)} \in (0, 1), \quad \varepsilon = e^{2a} > 1.$$

Analogous arguments applied to the second equation yield the estimate (1.5). Moreover, when both  $k$  and  $l$  are even, we obtain the equality

$$\|\Phi(k, l)P_l\| = K\alpha^{k-l}\varepsilon^l, \quad k \geq l,$$

which means that the nonuniform part  $\varepsilon^l = e^{2al}$  cannot be removed.

Although the notion of nonuniform exponential dichotomy has been studied in a very wide range and many rich results have been obtained, up to now there are no results on the spectral theory of (1.1) in the setting of nonuniform exponential dichotomies. In this paper, we establish the spectral theory in the setting of strong nonuniform exponential dichotomies. We say that (1.1) admits a *strong nonuniform exponential dichotomy* if it admits a nonuniform exponential dichotomy with  $\alpha\varepsilon^2 < 1$  in (1.4)-(1.5). For example, if  $\omega > 5a$ , then (1.6) admits a strong nonuniform exponential dichotomy. We remark that the phrase “strong nonuniform exponential dichotomy” has been used in [8], however here we use this notion in a different sense. Moreover, [7, Theorem 1.4.2] indicates that the condition  $\alpha\varepsilon^2 < 1$  is reasonable, which means that the constant  $\varepsilon$  belongs to the interval  $[1, \sqrt{1/\alpha})$ .

Among the different topics on classical exponential dichotomies, the dichotomy spectrum is very important and many results have been obtained. We refer the reader to [2, 3, 18, 26, 29, 30, 32, 34, 35] and the references therein. The definition and investigation for finite-time hyperbolicity has also been studied in [16, 22, 23].

This paper is organized as follows. In Section 2 we propose a definition of spectrum based on strong nonuniform exponential dichotomies, which is called nonuniform dichotomy spectrum. Such a spectrum can be seen as a generalization of Sacker-Sell spectrum. We prove a nonuniform dichotomy spectral theorem. In Section 3 we prove a reducibility result for (1.1) using the spectral result. Recall that system (1.1) is reducible if it is kinematically similar to a block diagonal system with blocks of dimension less than  $N$ .

## 2. NONUNIFORM DICHOTOMY SPECTRUM

Consider the weighted system

$$(2.1) \quad x_{k+1} = \frac{1}{\gamma} A_k x_k,$$

where  $\gamma \in \mathbb{R}^+ = (0, \infty)$ . One can easily see that

$$\Phi_\gamma(k, l) := \left(\frac{1}{\gamma}\right)^{k-l} \Phi(k, l)$$

is its evolution operator. If for some  $\gamma \in \mathbb{R}^+$ , (2.1) admits a nonuniform exponential dichotomy with projector  $P_k$  and constants  $K \geq 1, 0 < \alpha < 1$  and  $\varepsilon \geq 1$ , then  $P_k$  is also invariant for (1.1), that is

$$P_{k+1} A_k = A_k P_k, \quad k \in \mathbb{Z},$$

and the dichotomy estimates of (2.1) are equivalent to

$$(2.2) \quad \|\Phi(k, l) P_l\| \leq K(\gamma \alpha)^{k-l} \varepsilon^l, \quad k \geq l,$$

and

$$(2.3) \quad \|\Phi(k, l) Q_l\| \leq K(\gamma \frac{1}{\alpha})^{k-l} \varepsilon^l, \quad k \leq l.$$

**Definition 2.1.** *The nonuniform dichotomy spectrum of (1.1) is the set*

$\Sigma_{NED}(A) = \{\gamma \in \mathbb{R}^+ : (2.1) \text{ admits no strong nonuniform exponential dichotomy}\}$ ,  
and the resolvent set  $\rho_{NED}(A) = \mathbb{R}^+ \setminus \Sigma_{NED}(A)$  is its complement. The dichotomy spectrum of (1.1) is the set

$$\Sigma_{ED}(A) = \{\gamma \in \mathbb{R}^+ : (2.1) \text{ admits no exponential dichotomy}\},$$

and  $\rho_{ED}(A) = \mathbb{R}^+ \setminus \Sigma_{ED}(A)$ .

**Proposition 1.**  $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$ .

**Proof.** For each  $\gamma \in \rho_{ED}(A)$ , the weighted system (2.1) admits an exponential dichotomy. Consequently, the weighted system (2.1) admits a strong nonuniform exponential dichotomy. Thus,  $\gamma \in \rho_{NED}(A)$ , which implies that  $\rho_{ED}(A) \subset \rho_{NED}(A)$ , and therefore  $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$ .  $\square$

Let us define for  $\gamma \in \rho_{NED}(A)$

$$\mathcal{S}_\gamma := \{(l, \xi) \in \mathbb{Z} \times \mathbb{R}^N : \sup_{k \geq l} \|\Phi(k, l) \xi\| \gamma^{-k} \varepsilon^{-l} < \infty\},$$

and

$$\mathcal{U}_\gamma := \{(l, \xi) \in \mathbb{Z} \times \mathbb{R}^N : \sup_{k \leq l} \|\Phi(k, l) \xi\| \gamma^{-k} \varepsilon^{-l} < \infty\},$$

where  $\varepsilon$  is the constant in (2.2)-(2.3). One may readily verify that  $\mathcal{S}_\gamma$  and  $\mathcal{U}_\gamma$  are invariant vector bundles of (1.1), here we say that a nonempty set  $\mathcal{W} \subset \mathbb{Z} \times \mathbb{R}^N$  is an invariant vector bundle of (1.1) if (a) it is invariant, i.e.,  $(l, \xi) \in \mathcal{W} \Rightarrow (k, \Phi(k, l)\xi) \in \mathcal{W}$  for all  $k \in \mathbb{Z}$ ; and (b) for every  $l \in \mathbb{Z}$  the fiber  $\mathcal{W}(l) = \{\xi \in \mathbb{R}^N : (l, \xi) \in \mathcal{W}\}$  is a linear subspace of  $\mathbb{R}^N$ .

As a first glance,  $\mathcal{S}_\gamma$  and  $\mathcal{U}_\gamma$  are not well defined because they seem to depend on the constant  $\varepsilon$ , which may be not unique in (2.2)-(2.3). However, the following result ensures that  $\mathcal{S}_\gamma$  and  $\mathcal{U}_\gamma$  are well defined and they do not depend on the choice of the constant  $\varepsilon$ . First we recall that the invariant projector  $P$  is unique for (1.1) and (2.1) following the arguments in [21, Chapter 2]. Although the arguments in [21] are done in the setting of exponential dichotomies, it is not difficult to verify that they are also applicable to the case of nonuniform exponential dichotomies.

**Lemma 2.2.** *Assume that (2.1) admits a strong nonuniform exponential dichotomy with invariant projector  $P$  for  $\gamma \in \mathbb{R}^+$ . Then*

$$\mathcal{S}_\gamma = \text{im } P, \quad \mathcal{U}_\gamma = \ker P \quad \text{and} \quad \mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{Z} \times \mathbb{R}^N.$$

**Proof.** We show only  $\mathcal{S}_\gamma = \text{im } P$ . The fact  $\mathcal{U}_\gamma = \ker P$  is analog and the fact  $\mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{Z} \times \mathbb{R}^N$  is clear.

First we show  $\mathcal{S}_\gamma \subset \text{im } P$ . Let  $l \in \mathbb{Z}$  and  $\xi \in \mathcal{S}_\gamma(l)$ . Then there exists a positive constant  $C$  such that

$$\|\Phi(k, l)\xi\| \leq C\gamma^k \varepsilon^l, \quad k \geq l.$$

We write  $\xi = \xi_1 + \xi_2$  with  $\xi_1 \in \text{im } P_l$  and  $\xi_2 \in \ker P_l$ . We show that  $\xi_2 = 0$ . The invariance of  $P$  implies for  $k \in \mathbb{Z}$ , we have the identity

$$\xi_2 = \Phi_\gamma(l, k)\Phi_\gamma(k, l)Q_l\xi = \Phi_\gamma(l, k)Q_k\Phi_\gamma(k, l)\xi.$$

Since (2.1) admits a strong nonuniform exponential dichotomy, the following inequality holds

$$\|\Phi_\gamma(l, k)Q_k\| \leq K\left(\frac{1}{\alpha}\right)^{l-k} \varepsilon^k.$$

Thus

$$\begin{aligned} \|\xi_2\| &\leq K\left(\frac{1}{\alpha}\right)^{l-k} \varepsilon^k \|\Phi_\gamma(k, l)\xi\| \\ &= K(\alpha\varepsilon)^{k-l} \varepsilon^l \left(\frac{1}{\gamma}\right)^{k-l} \|\Phi(k, l)\xi\| \\ &\leq CK(\alpha\varepsilon)^{k-l} \varepsilon^{2l} \left(\frac{1}{\gamma}\right)^{k-l} \gamma^k \\ &= CK(\alpha\varepsilon)^{k-l} \varepsilon^{2l} \gamma^l \quad k \geq l, \end{aligned}$$

which implies that  $\xi_2 = 0$  by letting  $k \rightarrow \infty$ , since  $\alpha\varepsilon < 1$ .

Next we show  $\text{im } P \subset \mathcal{S}_\gamma$ . Let  $l \in \mathbb{Z}$  and  $\xi \in \text{im } P_l$ , i.e.,  $P_l\xi = \xi$ . The nonuniform exponential dichotomy implies that

$$\|\Phi_\gamma(k, l)\xi\| \leq K\alpha^{k-l} \varepsilon^l \|\xi\| \leq K\varepsilon^l \|\xi\|, \quad k \geq l,$$

since  $\alpha < 1$ , which implies that

$$\|\Phi(k, l)\xi\| \leq K\gamma^{k-l} \varepsilon^l \|\xi\|,$$

and hence  $\xi \in \mathcal{S}_\gamma(l)$ .  $\square$

**Lemma 2.3.** *The resolvent set is open, i.e., for every  $\gamma \in \rho_{NED}(A)$ , there exists a constant  $\beta = \beta(\gamma) \in (0, 1)$  such that  $(\beta\gamma, \frac{1}{\beta}\gamma) \subset \rho_{NED}(A)$ . Furthermore,*

$$\mathcal{S}_\zeta = \mathcal{S}_\gamma \quad \text{and} \quad \mathcal{U}_\zeta = \mathcal{U}_\gamma \quad \text{for} \quad \zeta \in (\beta\gamma, \frac{1}{\beta}\gamma).$$

**Proof.** Let  $\gamma \in \rho_{NED}(A)$ . Then (2.1) admits a strong nonuniform exponential dichotomy, i.e., the estimates (2.2)-(2.3) hold with an invariant projector  $P$ , constants  $K \geq 0$ ,  $0 < \alpha < 1$  and  $\varepsilon \geq 1$ . For  $\beta := \sqrt{\alpha} \in (0, 1)$  and  $\zeta \in (\beta\gamma, \frac{1}{\beta}\gamma)$  we have

$$\Phi_\zeta(k, l) = \left(\frac{\gamma}{\zeta}\right)^{k-l} \Phi_\gamma(k, l).$$

Now  $P$  is also an invariant projector for

$$x_{k+1} = \frac{1}{\zeta} A_k x_k.$$

Moreover, we have the estimates

$$\|\Phi_\zeta(k, l)P_l\| \leq K\left(\frac{\gamma}{\zeta}\alpha\right)^{k-l}\varepsilon^l \leq K\beta^{k-l}\varepsilon^l, \quad k \geq l,$$

and

$$\|\Phi_\zeta(k, l)Q_l\| \leq K\left(\frac{\gamma}{\zeta}\frac{1}{\alpha}\right)^{k-l}\varepsilon^l \leq K\left(\frac{1}{\beta}\right)^{k-l}\varepsilon^l, \quad k \leq l.$$

Hence  $\zeta \in \rho_{NED}(A)$ . Therefore,  $\rho_{NED}(A)$  is an open set. Using Lemma 2.2, we know that  $\mathcal{S}_\zeta = \mathcal{S}_\gamma$  and  $\mathcal{U}_\zeta = \mathcal{U}_\gamma$ .  $\square$

**Corollary 2.4.**  $\Sigma_{NED}(A)$  is a closed set.

Using the facts proved above, we can obtain the following result, whose proof is similar as [4, Lemma 2.2], and therefore we omit the proof here.

**Lemma 2.5.** Let  $\gamma_1, \gamma_2 \in \rho_{NED}(A)$  with  $\gamma_1 < \gamma_2$ . Then  $\mathcal{F} = \mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}$  is an invariant vector bundle which satisfies exactly one of the following two alternatives and the statements given in each alternative are equivalent:

- |  |   |
|--|---|
| Alternative I  | Alternative II  |
| (A) $\mathcal{F} = \mathbb{Z} \times \{0\}$ .  | (A') $\mathcal{F} \neq \mathbb{Z} \times \{0\}$ .                       |
| (B) $[\gamma_1, \gamma_2] \subset \rho_{NED}(A)$ .   | (B') There is a $\zeta \in (\gamma_1, \gamma_2) \cap \Sigma_{NED}(A)$ . |
| (C) $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$ .                                  | (C') $\dim \mathcal{S}_{\gamma_1} < \dim \mathcal{S}_{\gamma_2}$ .      |
| (D) $\mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_\gamma = \mathcal{U}_{\gamma_2}$<br>for $\gamma \in [\gamma_1, \gamma_2]$ . | (D') $\dim \mathcal{U}_{\gamma_1} > \dim \mathcal{U}_{\gamma_2}$ .      |

Now we are in a position to state and prove the nonuniform dichotomy spectral theorem which will be essential to prove the reducibility result in Section 3. The proof follows the idea and technique of the classical dichotomy spectrum proposed in [33], we present the details for the reader's convenience.

**Theorem 2.6.** The nonuniform dichotomy spectrum  $\Sigma_{NED}(A)$  of (1.1) is the disjoint union of  $n$  closed intervals (called spectral intervals) where  $0 \leq n \leq N$ , i.e.,  $\Sigma_{NED}(A) = \emptyset$  or  $\Sigma_{NED}(A) = \mathbb{R}^+$  or one of the four cases

$$\Sigma_{NED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\},$$

where  $0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ . Choose a

$$(2.4) \quad \gamma_0 \in \rho_{NED}(A) \text{ with } (0, \gamma_0) \subset \rho_{NED}(A) \text{ if possible,}$$

otherwise define  $\mathcal{U}_{\gamma_0} := \mathbb{Z} \times \mathbb{R}^N$ ,  $\mathcal{S}_{\gamma_0} := \mathbb{Z} \times \{0\}$ . Choose a

$$(2.5) \quad \gamma_n \in \rho_{NED}(A) \text{ with } (\gamma_n, +\infty) \subset \rho_{NED}(A) \text{ if possible,}$$

otherwise define  $\mathcal{U}_{\gamma_n} := \mathbb{Z} \times \{0\}$ ,  $\mathcal{S}_{\gamma_0} := \mathbb{Z} \times \mathbb{R}^N$ . Then the sets

$$\mathcal{W}_0 = \mathcal{S}_{\gamma_0} \quad \text{and} \quad \mathcal{W}_{n+1} = \mathcal{S}_{\gamma_n}$$

are invariant vector bundles of (1.1). For  $n \geq 2$ , choose  $\gamma_i \in \rho_{NED}(A)$  with

$$(2.6) \quad b_i < \gamma_i < a_{i+1} \quad \text{for } i = 1, \dots, n-1,$$

then for every  $i = 1, \dots, n-1$  the intersection

$$\mathcal{W}_i = \mathcal{U}_{\gamma_{i-1}} \cap \mathcal{S}_{\gamma_i}$$

is an invariant vector bundle of (1.1) with  $\dim \mathcal{W}_i \geq 1$ . The invariant vector bundles  $\mathcal{W}_i, i = 0, \dots, n+1$ , are called spectral bundles and they are independent of the choice of  $\gamma_0, \dots, \gamma_n$  in (2.4), (2.5) and (2.6). Moreover

$$\mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$$

is a Whitney sum, i.e.,  $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{Z} \times \{0\}$  for  $i \neq j$  and  $\mathcal{W}_0 + \dots + \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$ .

**Proof.** Recall that the resolvent set  $\rho_{NED}(A)$  is open and therefore  $\Sigma_{NED}(A)$  is the disjoint union of closed intervals. Next we will show that  $\Sigma_{NED}(A)$  consists of at most  $N$  intervals. Indeed, if  $\Sigma_{NED}(A)$  contains  $N+1$  components, then one can choose a collections of points  $\zeta_1, \dots, \zeta_N$  in  $\rho_{NED}(A)$  such that  $\zeta_1 < \dots < \zeta_N$  and each of the intervals  $(0, \zeta_1), (\zeta_1, \zeta_2), \dots, (\zeta_{N-1}, \zeta_N), (\zeta_N, \infty)$  has nonempty intersection with the spectrum  $\Sigma_{NED}(A)$ . Now alternative II of Lemma 2.5 implies

$$0 \leq \dim \mathcal{S}_{\zeta_1} < \dots < \dim \mathcal{S}_{\zeta_N} \leq N$$

and therefore either  $\dim \mathcal{S}_{\zeta_1} = 0$  or  $\dim \mathcal{S}_{\zeta_N} = N$  or both. Without loss of generality,  $\dim \mathcal{S}_{\zeta_N} = N$ , i.e.,  $\mathcal{S}_{\zeta_N} = \mathbb{Z} \times \mathbb{R}^N$ . Assume that

$$x_{k+1} = \frac{1}{\zeta_N} A_k x_k$$

admits a strong nonuniform exponential dichotomy with invariant projector  $P \equiv \text{Id}$ , then

$$x_{k+1} = \frac{1}{\zeta} A_k x_k$$

also admits a strong nonuniform exponential dichotomy with the same projector for every  $\zeta > \zeta_N$ . Now we have the conclusion  $(\zeta_N, \infty) \subset \rho_{NED}(A)$ , which is a contradiction. This proves the alternatives for  $\Sigma_{NED}(A)$ .

Due to Lemma 2.5, the sets  $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$  are invariant vector bundles. To prove now that  $\dim \mathcal{W}_1 \geq 1, \dots, \dim \mathcal{W}_n \geq 1$  for  $n \geq 1$ , let us assume that  $\dim \mathcal{W}_1 = 0$ , i.e.,  $\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1} = \mathbb{Z} \times \{0\}$ . If  $(0, b_1]$  is a spectral interval this implies that  $\mathcal{S}_{\gamma_1} = \mathbb{Z} \times \{0\}$ . Then the projector of the nonuniform exponential dichotomy of

$$x_{k+1} = \frac{1}{\gamma_1} A_k x_k$$

is 0 and then we get the contraction  $(0, \gamma_1) \subset \rho_{NED}(A)$ . If  $[a_1, b_1]$  is a spectral interval then  $[\gamma_0, \gamma_1] \cap \Sigma_{NED}(A) \neq \emptyset$  and alternative II of Lemma 2.5 yields a contradiction. Therefore  $\dim \mathcal{W}_1 \geq 1$  and similarly  $\dim \mathcal{W}_n \geq 1$ . Furthermore for  $n \geq 3$  and  $i = 2, \dots, n-1$  one has  $[\gamma_{i-1}, \gamma_i] \cap \Sigma_{NED}(A) \neq \emptyset$  and again alternative II of Lemma 2.5 yields  $\dim \mathcal{W}_i \geq 1$ .

For  $i < j$  we have  $\mathcal{W}_i \subset \mathcal{S}_{\gamma_i}$  and  $\mathcal{W}_j \subset \mathcal{U}_{\gamma_{j-1}} \subset \mathcal{U}_{\gamma_i}$  and with Lemma 2.2 this gives  $\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{S}_{\gamma_i} \cap \mathcal{U}_{\gamma_i} = \mathbb{Z} \times \{0\}$ , so  $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{Z} \times \{0\}$  for  $i \neq j$ .

To show that  $\mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$ , recall the monotonicity relations  $\mathcal{S}_{\gamma_0} \subset \dots \subset \mathcal{S}_{\gamma_n}, \mathcal{U}_{\gamma_0} \supset \dots \supset \mathcal{U}_{\gamma_n}$ , and the identity  $\mathcal{S}_{\gamma} \oplus \mathcal{U}_{\gamma} = \mathbb{Z} \times \mathbb{R}^N$  for  $\gamma \in \mathbb{R}^+$ .

Therefore  $\mathbb{Z} \times \mathbb{R}^N = \mathcal{W}_0 \times \mathcal{U}_{\gamma_0}$ . Now we have

$$\begin{aligned}\mathbb{Z} \times \mathbb{R}^N &= \mathcal{W}_0 + \mathcal{U}_{\gamma_0} \cap [\mathcal{S}_{\gamma_1} + \mathcal{U}_{\gamma_1}] \\ &= \mathcal{W}_0 + [\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}] + \mathcal{U}_{\gamma_1} \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{U}_{\gamma_1}.\end{aligned}$$

Doing the same for  $\mathcal{U}_{\gamma_1}$ , we get

$$\begin{aligned}\mathbb{Z} \times \mathbb{R}^N &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{U}_{\gamma_1} \cap [\mathcal{S}_{\gamma_2} + \mathcal{U}_{\gamma_2}] \\ &= \mathcal{W}_0 + \mathcal{W}_1 + [\mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}] + \mathcal{U}_{\gamma_2} \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{U}_{\gamma_2},\end{aligned}$$

and mathematical induction yields  $\mathbb{Z} \times \mathbb{R}^N = \mathcal{W}_0 + \dots + \mathcal{W}_{n+1}$ . To finish the proof, let  $\tilde{\gamma}_0, \dots, \tilde{\gamma}_n \in \rho_{NED}(A)$  be given with the properties (2.4), (2.5) and (2.6). Then alternative I of Lemma 2.5 implies

$$\mathcal{S}_{\gamma_i} = \mathcal{S}_{\tilde{\gamma}_i} \quad \text{and} \quad \mathcal{U}_{\gamma_i} = \mathcal{U}_{\tilde{\gamma}_i} \quad \text{for} \quad i = 0, \dots, n$$

and therefore the invariant vector bundles  $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$  are independent of the choice of  $\gamma_0, \dots, \gamma_n$  in (2.4), (2.5) and (2.6).  $\square$

**Definition 2.7.** We say that (1.1) is nonuniformly exponentially bounded if there exist constants  $K > 0, \varepsilon \geq 1$  and  $a \geq 1$  such that

$$(2.7) \quad \|\Phi(k, l)\| \leq K a^{|k-l|} \varepsilon^l, \quad k, l \in \mathbb{Z}.$$

**Lemma 2.8.** Assume that (1.1) is nonuniformly exponentially bounded. Then  $\Sigma_{NED}(A)$  is a bounded closed set and  $\Sigma_{NED}(A) \subset [\frac{1}{a}, a]$ .

**Proof.** Assume that (2.7) holds. Let  $\gamma > a$  and  $0 < \alpha := \frac{a}{\gamma} < 1$ , then estimate (2.7) implies

$$\|\Phi_\gamma(k, l)\| \leq K \alpha^{k-l} \varepsilon^l, \quad k \geq l.$$

Therefore (1.1) admits a nonuniform exponential dichotomy with invariant projector  $P = I$ . We have  $\gamma \in \rho_{NED}(A)$  and similarly for  $0 < \gamma < \frac{1}{a}$ , therefore  $\Sigma_{NED}(A) \subset [\frac{1}{a}, a]$ .  $\square$

**Corollary 2.9.** If (1.1) is nonuniformly exponentially bounded, then the nonuniform dichotomy spectrum  $\Sigma_{NED}(A)$  of (1.1) is the disjoint union of  $n$  closed intervals where  $0 \leq n \leq N$ , i.e.,

$$\Sigma_{NED}(A) = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_{n-1}, b_{n-1}] \cup [a_n, b_n],$$

where  $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n$ .

From Proposition 1, we know  $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$ . Finally in this Section, we present an example to illustrate that  $\Sigma_{NED}(A) \neq \Sigma_{ED}(A)$  can occur.

**Example 2.10.** Given  $\omega > 5a > 0$ . Consider the scalar equation

$$(2.8) \quad u_{k+1} = A_k u_k$$

with

$$A_k = e^{-\omega + ak(-1)^k - a(k-1)(-1)^{(k-1)}}.$$

Then  $\Sigma_{NED}(A) = [e^{-\omega-a}, e^{-\omega+a}]$  and  $\Sigma_{ED}(A) = \mathbb{R}^+$ .

**Proof.** The evolution operator of (2.8) is given by

$$\Phi(k, l) = e^{-\omega(k-l-1)-a(k-l-1)(-1)^{k-1}-al(-1)^{(k-1)}+al(-1)^l}.$$

For any  $\gamma \in \mathbb{R}^+$  the evolution operator of the equation

$$(2.9) \quad u_{k+1} = \frac{1}{\gamma} A_k u_k$$

is given by

$$(2.10) \quad \Phi_\gamma(k, l) = \left(\frac{1}{\gamma}\right)^{(k-l)} e^{-\omega(k-l-1)-a(k-l-1)(-1)^{k-1}-al(-1)^{(k-1)}+al(-1)^l}.$$

For any  $\gamma \in (e^{(-\omega+5a)}, +\infty)$ , it follows from (2.10) that

$$(2.11) \quad |\Phi_\gamma(k, l)| \leq e^{\omega-a} \left(\frac{e^{-\omega+a}}{\gamma}\right)^{k-l} e^{2al}, \quad k \geq l,$$

which implies that the equation (2.9) admits a strong nonuniform exponential dichotomy with  $P = \text{Id}$ , by taking

$$K = e^{\omega-a}, \quad \alpha = \frac{e^{-\omega+a}}{\gamma} < 1, \quad \varepsilon = e^{2a} > 0.$$

Thus,

$$(2.12) \quad (e^{-\omega+5a}, +\infty) \subset \rho_{NED}(A).$$

For any  $\tilde{\gamma} \in (0, e^{-\omega-5a})$ , it follows from (2.10) that

$$(2.13) \quad |\Phi_\gamma(k, l)| \leq e^{\omega+a} \left(\frac{e^{-\omega-a}}{\gamma}\right)^{k-l} e^{2al}, \quad k \leq l,$$

which implies that (2.9) admits a strong nonuniform exponential dichotomy with  $P = 0$ , by taking

$$K = e^{\omega+a}, \quad \alpha = \frac{\gamma}{e^{-\omega-a}} < 1, \quad \varepsilon = e^{2a} > 0.$$

Thus,

$$(2.14) \quad (0, e^{-\omega-5a}) \subset \rho_{NED}(A).$$

It follows from (2.12) and (2.14) that

$$(0, e^{-\omega-5a}) \cup (e^{-\omega+5a}, +\infty) \subset \rho_{NED}(A),$$

which implies that

$$\Sigma_{NED}(A) \subset [e^{-\omega-5a}, e^{-\omega+5a}].$$

Next we show that

$$[e^{-\omega-5a}, e^{-\omega+5a}] \subset \Sigma_{NED}(A).$$

To do this, we first prove that  $\gamma_1 = e^{-\omega+5a} \in \Sigma_{NED}(A)$ . The evolution operator of the system

$$u_{k+1} = \frac{1}{\gamma_1} A_k u_k$$

is given as

$$\Phi_{\gamma_1}(k, l) = e^{\omega-a} e^{-a(k-l-1)(1+(-1)^{k-1})-al(-1)^{(k-1)}+al(-1)^l}.$$

It is easy to see that there do not exist  $K, \alpha > 0$  and  $\varepsilon > 0$  such that

$$\|\Phi_{\gamma_1}(k, l)\| \leq K \alpha^{k-l} \varepsilon^l, \quad \text{for } k \geq l,$$

or

$$\|\Phi_{\gamma_1}(k, l)\| \leq K \left(\frac{1}{\alpha}\right)^{k-l} \varepsilon^l, \quad \text{for } k \leq l.$$



Therefore  $\gamma_1 = e^{-\omega+5a} \in \Sigma_{NED}(A)$ . In a similar manner, we can prove  $\gamma_2 = e^{-\omega-5a} \in \Sigma_{NED}(A)$ . We can see from Theorem 2.6 that (2.8) has at most one nonuniform dichotomy spectral interval, which means that  $[e^{-\omega-5a}, e^{-\omega+5a}] \subset \Sigma_{NED}(A)$  and therefore  $[e^{-\omega-5a}, e^{-\omega+5a}] = \Sigma_{NED}(A)$ .

On the other hand, using a similar argument as in equations (1.6), we know that the nonuniform part  $\varepsilon^l$  cannot be removed in the estimates (2.11) and (2.13). Therefore, (2.8) does not admit an exponential dichotomy, which means that  $\Sigma_{ED}(A) = \mathbb{R}^+$ .  $\square$

### 3. REDUCIBILITY

In this section we employ Theorem 2.6 to prove a reducibility result. For the reducibility results in the setting of an exponential dichotomy, we refer the reader to [20, 27, 35] and the references therein.

**Lemma 3.1.** *The projector of equation (1.1) can be chosen as  $\tilde{P} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix}$  with  $N_1 = \dim \operatorname{im} \tilde{P}$  and  $N_2 = \dim \ker \tilde{P}$ , and the fundamental matrix  $X_k$  can be chosen suitably such that the estimates (1.4)-(1.5) can be rewritten as*

$$(3.1) \quad \|X_k \tilde{P} X_l^{-1}\| \leq K \alpha^{k-l} \varepsilon^l, \quad k \geq l,$$

and

$$(3.2) \quad \|X_k \tilde{Q} X_l^{-1}\| \leq K \left(\frac{1}{\alpha}\right)^{k-l} \varepsilon^l, \quad k \leq l,$$

where  $\tilde{Q} = \operatorname{Id} - \tilde{P}$ .

**Proof.** Let  $n \in \mathbb{Z}$  be arbitrary but fixed. Note that the rank of the projector  $P_n$  is independent of  $n \in \mathbb{Z}$  (see [16, Page 1100]), then there exists a nondegenerate matrix  $T \in \mathbb{R}^{N \times N}$  such that

$$\tilde{P} := \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix} = T P_n T^{-1}$$

with  $N_1 = \dim \operatorname{im} \tilde{P}$  and  $N_2 = \dim \ker \tilde{P}$ . Define

$$X_k := \Phi(k, n) T^{-1} \quad \text{for } k \in \mathbb{Z} \quad \text{and} \quad \tilde{P} := \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix} = T P_n T^{-1}.$$

Then

$$(3.3) \quad \|X_k \tilde{P} X_l^{-1}\| = \|\Phi(k, n) T^{-1} \tilde{P} T \Phi^{-1}(l, n)\| = \|\Phi(k, n) P_n \Phi^{-1}(l, n)\|.$$

On the other hand, we have

$$(3.4) \quad \begin{aligned} \|\Phi(k, l) P_l\| &= \|\Phi(k, n) \Phi(n, l) P_l\| \\ &= \|\Phi(k, n) P_n \Phi(n, l)\| \\ &= \|\Phi(k, n) P_n \Phi^{-1}(l, n)\|. \end{aligned}$$

It follows from (3.3) and (3.4) that (1.4)-(1.5) can be rewritten in the form (3.1)-(3.2).  $\square$

Now we recall the definition of kinematic similarity and several results in Coppel [21] and Aulbach et al. [2].

**Definition 3.2.** Equation (1.1) is said to be kinematically similar to another equation

$$(3.5) \quad y_{k+1} = B_k y_k$$

with  $k \in \mathbb{Z}$ , if there exists an invertible matrix  $S_k$  with  $\|S_k\| \leq M$  and  $\|S_k^{-1}\| \leq M$  ( $M > 0$ ), which satisfies the difference equation

$$S_{k+1}B_k = A_kS_k.$$

The change of variables  $x_k = S_k y_k$  then transforms (1.1) into (3.5).

The next lemma is important to establish the reducibility results and its proof follows along the lines of the proof of Siegmund [35]. See also Coppel [21] and Aulbach et al. [2]

**Lemma 3.3.** [21, Chapter 5] Let  $P$  be an orthogonal projection ( $P^T = P$ ) and let  $X$  be an invertible matrix. Then there exists an invertible matrix function  $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$  such that

$$S_k P S_k^{-1} = X_k P X_k^{-1}, \quad S_k Q S_k^{-1} = X_k Q X_k^{-1},$$

and

$$\|S_k\| \leq \sqrt{2},$$

$$\|S_k^{-1}\| \leq [\|X_k P X_k^{-1}\|^2 + \|X_k (I - P) X_k^{-1}\|^2]^{\frac{1}{2}},$$

where  $k \in \mathbb{Z}$  and  $Q = \text{Id} - P$ . Define

$$\tilde{R} : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}, \quad k \mapsto P X_k^T X_k P + [\text{Id} - P] X_k^T X_k [\text{Id} - P].$$

Then the mapping is a positive definite, symmetric matrix for every  $k \in \mathbb{Z}$ . Moreover there is a unique function

$$R : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$$

of positive definite symmetric matrices  $R_k$ ,  $k \in \mathbb{Z}$ , with

$$R_k^2 = \tilde{R}_k, \quad P R_k = R_k P.$$

We remark that  $S_k^{-1}$  in Lemma 3.3 is bounded in the setting of an exponential dichotomy. However, in the setting of a nonuniform exponential dichotomy,  $S_k^{-1}$  can be unbounded, because  $\|\Phi(k, k)P_k\| \leq K\varepsilon^k$  for  $k \geq 0$ . To overcome the difficulty, we introduce a new version of non-degeneracy, so-called weak non-degeneracy and define the concept of weak kinematical similarity. Some results will be obtained on the decoupling into two blocks which will play an important role in the analysis of reducibility.

**Definition 3.4.**  $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$  is called weakly non-degenerate if there exists a constant  $M = M(\varepsilon) > 0$  such that

$$\|S_k\| \leq M\varepsilon^{|k|} \quad \text{and} \quad \|S_k^{-1}\| \leq M\varepsilon^{|k|}, \quad \text{for all } k \in \mathbb{Z}.$$

**Definition 3.5.** If there exists a weakly non-degenerate matrix  $S_k$  such that

$$S_{k+1}B_k = A_kS_k,$$

then equation (1.1) is weakly kinematically similar to equation (3.5). For short, we denote (1.1)  $\overset{w}{\sim}$  (3.5) or  $A_k \overset{w}{\sim} B_k$ .

For the sake of comparison, we denote kinematical similarity by (1.1)  $\sim$  (3.5) or  $A_k \sim B_k$ .

**Definition 3.6.** We say that equation (1.1) is reducible, if it is weakly kinematically similar to equation (3.5) whose coefficient matrix  $B_k$  has the block form

$$(3.6) \quad \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix},$$

where  $B_k^1$  and  $B_k^2$  are matrices of smaller size than  $B_k$ .

The following theorem shows that if (1.1) admits a nonuniform exponential dichotomy, then there exists a weakly non-degenerate transformation such that  $A_k \overset{w}{\sim} B_k$  and  $B_k$  has the block form (3.6), i.e., system (1.1) is reducible.

**Theorem 3.7.** Assume that (1.1) admits a nonuniform exponential dichotomy (not necessary strong) of the form (3.1)-(3.2) with invariant projector  $P_k \neq 0, \text{Id}$ . Then (1.1) is weakly kinematically similar to a decoupled system

$$(3.7) \quad x_{k+1} = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix} x_k$$

for some locally integrable matrix functions

$$B^1 : \mathbb{Z} \rightarrow \mathbb{R}^{N_1 \times N_1} \quad \text{and} \quad B^2 : \mathbb{Z} \rightarrow \mathbb{R}^{N_2 \times N_2}$$

where  $N_1 := \dim \text{im } \tilde{P}$  and  $N_2 := \dim \ker \tilde{P}$ . That is, system (1.1) is reducible.

**Proof.** Since equation (1.1) admits a nonuniform exponential dichotomy of the form (1.4)-(1.5) with invariant projector  $P_k \neq 0, \text{Id}$ , by Lemma 3.1, we can choose suitable fundamental matrix  $X_k$  and the projector  $\tilde{P} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0 \end{pmatrix}$ , ( $0 < N_1 < N$ ) such that the estimates (3.1)-(3.2) hold. By Lemma 3.3 and the estimates (3.1)-(3.2), there exists a  $M = M(\varepsilon) > 0$  large enough such that

$$\|S_k\| \leq \sqrt{2} \leq M\varepsilon^{|k|},$$

$$\|S_k^{-1}\| \leq [\|X_k \tilde{P} X_k^{-1}\|^2 + \|X_k(I - \tilde{P})X_k^{-1}\|^2]^{\frac{1}{2}} \leq \sqrt{2}K\varepsilon^{|k|}.$$

Thus,  $S$  is weakly non-degenerate. Setting

$$B_k = R_{k+1} R_k^{-1},$$

where  $R_k$  is defined in Lemma 3.3 and  $X_k = S_k R_k$ . Obviously,  $R_k$  is the fundamental matrix of linear system

$$y_{k+1} = B_k y_k.$$

Now we need to show that  $A_k \overset{w}{\sim} B_k$  and  $B_k$  has the block diagonal form

$$B_k = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix}, \quad \text{for } k \in \mathbb{Z}.$$

First, we show that  $A_k \overset{w}{\sim} B_k$ . In fact,

$$\begin{aligned} S_{k+1} B_k &= X_{k+1} R_{k+1}^{-1} B_k \\ &= A_k X_k R_k^{-1} B_k^{-1} B_k \\ &= A_k S_k, \end{aligned}$$

which implies that  $A_k \overset{w}{\sim} B_k$ .

Now we show that system (1.1) is weakly kinematically similar to (3.7). By Lemma 3.3,  $R_{k+1}$  and  $R_k^{-1}$  commute with the matrix  $\tilde{P}$  for every  $k \in \mathbb{Z}$ . It follows that

$$(3.8) \quad \tilde{P}B_k = B_k\tilde{P}$$

for all  $k \in \mathbb{Z}$ . Now we decompose  $B_k : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$  into four functions

$$\begin{aligned} B_k^1 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_1 \times N_1}, & B_k^2 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_2 \times N_2}, \\ B_k^3 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_1 \times N_2}, & B_k^4 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_2 \times N_1}, \end{aligned}$$

with

$$B_k = \begin{pmatrix} B_k^1 & B_k^3 \\ B_k^4 & B_k^2 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Identity (3.8) implies that

$$\begin{pmatrix} B_k^1 & B_k^3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_k^1 & 0 \\ B_k^4 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Therefore  $B_k^3 \equiv 0$  and  $B_k^4 \equiv 0$ . Thus  $B_k$  has the block form

$$B_k = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Now the proof is finished.  $\square$

From Theorem 3.7, we know that if (1.1) admits a nonuniform exponential dichotomy, then there exists a weakly non-degenerate transformation  $S_k$  such that  $A_k \stackrel{w}{\sim} B_k$  and  $B_k$  has two blocks of the form (3.6).

**Lemma 3.8.** *Assume that (1.1) admits a nonuniform exponential dichotomy with the form of estimates (3.1)-(3.2) and  $\text{rank}(\tilde{P}) = N_1$ , ( $0 < N_1 < N$ ), and there exists a weakly non-degenerate transformation  $S_k$  such that  $A_k \stackrel{w}{\sim} B_k$ . Then system (3.5) also admits a nonuniform exponential dichotomy, and the projector has the same rank.*

**Proof.** Suppose that  $S_k$  is weakly non-degenerate, which means that there exists  $M = M(\varepsilon) > 0$  such that  $\|S_k\| \leq M\varepsilon^{|k|}$  and  $\|S_k^{-1}\| \leq M\varepsilon^{|k|}$  and such that  $A_k \stackrel{w}{\sim} B_k$ . Let  $X_k = S_k Y_k$ . It is easy to see that  $Y_k$  is the fundamental matrix of system (3.5). To prove that system (3.5) admits a nonuniform exponential dichotomy, we first consider the case  $k \geq l$  and obtain

$$(3.9) \quad \begin{aligned} \|Y_k \tilde{P} Y_l^{-1}\| &= \|S_k^{-1} X_k \tilde{P} X_l^{-1} S_k\| \\ &\leq \|S_k^{-1}\| \cdot \|X_k \tilde{P} X_l^{-1}\| \cdot \|S_l\| \\ &\leq K M^2 \varepsilon^{|k|} \alpha^{k-l} \varepsilon^l \varepsilon^{|l|} \\ &\leq K M_1^2 (\varepsilon \alpha)^{k-l} \varepsilon^l, \quad k \geq l, \end{aligned}$$

where  $M_1 = M\varepsilon^{2|l|}$ . Similar argument shows that

$$(3.10) \quad \|Y_k \tilde{Q} Y_l^{-1}\| \leq K M_1^2 \left(\frac{1}{\varepsilon \alpha}\right)^{k-l} \varepsilon^l, \quad k \leq l.$$

Form (3.9) and (3.10), it is easy to see that system (3.5) admits a nonuniform exponential dichotomy. Clearly, the rank of the projector is  $k$ .  $\square$

**Lemma 3.9.** *Assume that the systems (1.1) and (3.5) are weakly kinematically similar via  $S_k$ . If for a constant  $\gamma \in \mathbb{R}^+$  the system (2.1) admits a strong nonuniform exponential dichotomy with constants  $K > 0$ ,  $0 < \alpha < 1$ ,  $\varepsilon \geq 1$  and invariant projector  $P$ , then the system*

$$(3.11) \quad y_{k+1} = \frac{1}{\gamma} B_k y_k$$

*also admits a strong nonuniform exponential dichotomy.*

**Proof.** Obviously,  $P$  is also an invariant projector for (1.1). The dichotomy estimates are equivalent to

$$\|X_k P X_l^{-1}\| \leq K \alpha^{k-l} \varepsilon^l, \quad k \geq l,$$

and

$$\|X_k P X_l^{-1}\| \leq K \left(\frac{1}{\alpha}\right)^{k-l} \varepsilon^l, \quad k \leq l.$$

Using Lemma 3.8, it is easy to see that

$$\|Y_k P Y_l^{-1}\| \leq K'_\gamma (\varepsilon \alpha)^{k-l} \varepsilon^l, \quad k \geq l,$$

and

$$\|Y_k P Y_l^{-1}\| \leq K'_\gamma \left(\frac{1}{\varepsilon \alpha}\right)^{k-l} \varepsilon^l, \quad k \leq l,$$

for some constant  $K'_\gamma \geq 1$ . Therefore, (3.11) admits a strong nonuniform exponential dichotomy.  $\square$

The following result follows directly from Lemma 3.9.

**Corollary 3.10.** *Assume that there exists a weakly non-degenerate transformation  $S_k$  such that  $A_k \stackrel{w}{\sim} B_k$ . Then  $\Sigma_{NED}(A) = \Sigma_{NED}(B)$ , i.e.,*

$$\Sigma_{NED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\} = \Sigma_{NED}(B).$$

Now we are in a position to prove the reducibility result.

**Theorem 3.11** (Reducibility Theorem). *Assume that (1.1) admits a strong nonuniform exponential dichotomy. Due to Theorem 2.6, the dichotomy spectrum is either empty or the disjoint union of  $n$  closed spectral intervals  $\mathcal{I}_1, \dots, \mathcal{I}_n$  with  $1 \leq n \leq N$ , i.e.,*

$$\Sigma_{NED}(A) = \emptyset \quad (n = 0) \quad \text{or} \quad \Sigma_{NED}(A) = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n.$$

*Then there exists a weakly kinematic similarity action  $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$  between (1.1) and a block diagonal system*

$$x_{k+1} = \begin{pmatrix} B_k^0 & & \\ & \ddots & \\ & & B_k^{n+1} \end{pmatrix} x_k$$

*with  $B^i : \mathbb{Z} \rightarrow \mathbb{R}^{N_i \times N_i}$ ,  $N_i = \dim \mathcal{W}_i$ , and*

$$\Sigma_{NED}(B^0) = \emptyset, \Sigma_{NED}(B^1) = \mathcal{I}_1, \dots, \Sigma_{NED}(B^n) = \mathcal{I}_n, \Sigma_{NED}(B^{n+1}) = \emptyset.$$

**Proof.** If for any  $\gamma \in \mathbb{R}^+$ , system (2.1) admits a strong nonuniform exponential dichotomy, then  $\Sigma_{NED}(A) = \emptyset$ . Conversely, for any  $\gamma \in \mathbb{R}^+$ , system (2.1) does not admit a strong nonuniform exponential dichotomy, then  $\Sigma_{NED}(A) = \mathbb{R}^+$ . Now, we prove the theorem for the nontrivial case ( $\Sigma_{NED}(A) \neq \emptyset$  and  $\Sigma_{NED}(A) \neq \mathbb{R}^+$ ).

Recall that the resolvent set  $\rho_{NED}(A)$  is open and therefore the dichotomy spectrum  $\Sigma_{NED}(A)$  is the disjoint union of closed intervals. Using Theorem 2.6, we can assume

$$\mathcal{I}_1 = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\}, \mathcal{I}_2 = [a_2, b_2], \dots, \mathcal{I}_{n-1} = [a_{n-1}, b_{n-1}], \mathcal{I}_n = \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\}$$

with  $0 < a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n$ .

If  $\mathcal{I}_1 = [a_1, b_1]$  is a spectral interval, then  $(0, \gamma_0) \subset \rho_{NED}(A)$  and  $\mathcal{W}_0 = \mathcal{S}_{\gamma_0}$  for some  $\gamma_0 < a_1$  due to Theorem 2.6, which implies that

$$x_{k+1} = \frac{1}{\gamma_0} A_k x_k$$

admits a strong nonuniform exponential dichotomy with an invariant projector  $\tilde{P}_0$ . By Theorem 3.7 and Corollary 3.10, there exists a weakly non-degenerate transformation  $x_k = S_k^0 x_k^{(0)}$  with  $\|S_k^0\| \leq M_0 \varepsilon^{|k|}$  and  $\|(S_k^0)^{-1}\| \leq M_0 \varepsilon^{|k|}$  for some positive constant  $M_0 = M_0(\varepsilon)$  and such that  $A_k \stackrel{w}{\sim} A_k^0$  and  $A_k^0$  has two blocks of the form  $A_k^0 = \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix}$  with  $\dim B_k^0 = \dim \text{im } \tilde{P}_0 = \dim \mathcal{S}_{\gamma_0} = \dim \mathcal{W}_0 =: N_0$  due to Theorem 3.7, Lemma 2.2 and Theorem 2.6. If  $\mathcal{I}_1 = (0, b_1]$  is a spectral interval, a block  $B_k^0$  is omitted.

Now we consider the following system

$$x_{k+1}^{(0)} = A_k^0 x_k^{(0)} = \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix} x_k^{(0)}.$$

By using Lemma 2.5, we take  $\gamma_1 \in (b_1, a_2)$ . In view of  $(b_1, a_2) \subset \rho_{NED}(B_k^{0,*})$ ,  $\gamma_1 \in \rho_{NED}(B_k^{0,*})$ , which implies that

$$x_{k+1}^{(0)} = \frac{1}{\gamma_1} \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix} x_k^{(0)}$$

admits a nonuniform exponential dichotomy with an invariant projector  $\tilde{P}_1$ . From the claim above, we know that  $\tilde{P}_1 \neq 0, I$ . Similarly by Theorem 3.7 and Corollary 3.10, there exists a weakly non-degenerate transformation

$$x_k^{(0)} = S_k^1 x_k^{(1)} = \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix} x_k^{(1)}$$

with  $\|\tilde{S}_k^1\| \leq M_1 \varepsilon^{|k|}$  and  $\|(\tilde{S}_k^1)^{-1}\| \leq M_1 \varepsilon^{|k|}$  for some positive constant  $M_1 = M_1(\varepsilon)$  and such that  $B_k^{0,*} \stackrel{w}{\sim} \tilde{B}_k^{0,*}$  and  $\tilde{B}_k^{0,*}$  has two blocks of the form  $\tilde{B}_k^{0,*} = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^{1,*} \end{pmatrix}$  with  $\dim B_k^1 = \dim \text{im } \tilde{P}_1 = \dim \mathcal{S}_{\gamma_1} \geq \dim(\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}) = \dim \mathcal{W}_1 =: N_1$  due to Theorem 3.7, Lemma 2.2 and Theorem 2.6. In addition, using Theorem 3.7 and Corollary 3.10, we have

$$\Sigma_{NED}(B_k^1) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\}, \Sigma_{NED}(B_k^{1,*}) = [a_2, b_2] \cup \dots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\}.$$

Now we can construct a weakly non-degenerate transformation  $x_k = \tilde{S}_k x_k^{(1)}$  with  $\tilde{S}_k = S_k^0 S_k^1 = S_k^0 \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix}$ , where  $\|\tilde{S}_k\| \leq M_0 M_1 \varepsilon^{2|k|}$  and  $\|\tilde{S}_k^{-1}\| \leq M_0 M_1 \varepsilon^{2|k|}$ .

Then  $A_k \stackrel{w}{\sim} A_k^1$  and  $A_k^1$  has three blocks of the form

$$A_k^1 = \begin{pmatrix} B_k^0 & & \\ & B_k^1 & \\ & & B_k^{1,*} \end{pmatrix}.$$

Applying similar procedures to  $\gamma_2 \in (b_2, a_3)$ ,  $\gamma_3 \in (b_3, a_4), \dots$ , we can construct a weakly non-degenerate transformation  $x_k = S_k x_k^{(n+1)}$  with

$$S_k = S_k^0 \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix} \begin{pmatrix} I_{N_0+N_1} & 0 \\ 0 & \tilde{S}_k^2 \end{pmatrix} \cdots \begin{pmatrix} I_{N_0+\dots+N_{n-1}} & 0 \\ 0 & \tilde{S}_k^n \end{pmatrix}$$

such that  $\|S_k\| \leq M_\varepsilon \varepsilon^{n|k|}$  and  $\|S_k^{-1}\| \leq M_\varepsilon \varepsilon^{n|k|}$  with  $M_\varepsilon = M_0 \times \dots \times M_n$ . Now we can prove

$$A_k \stackrel{w}{\sim} A_k^n := B_k = \begin{pmatrix} B_k^0 & & \\ & \ddots & \\ & & B_k^{n+1} \end{pmatrix}$$

with locally integrable functions  $B^i : \mathbb{Z} \rightarrow \mathbb{R}^{N_i \times N_i}$  and

$$\Sigma_{NED}(B^0) = \emptyset, \Sigma_{NED}(B^1) = \mathcal{I}_1, \dots, \Sigma_{NED}(B^n) = \mathcal{I}_n, \Sigma_{NED}(B^{n+1}) = \emptyset.$$

Finally, we show that  $N_i = \dim \mathcal{W}_i$ . From the claim above, we note that  $\dim B_k^0 = \dim \mathcal{W}_0$ ,  $\dim B_k^1 \geq \dim \mathcal{W}_1, \dots, \dim B_k^n \geq \dim \mathcal{W}_n$ ,  $\dim B_k^{n+1} = \dim \mathcal{W}_{n+1}$  and with Theorem 2.6 this gives  $\dim \mathcal{W}_0 + \dots + \dim \mathcal{W}_{n+1} = N$ , so  $\dim B_k^i = \dim \mathcal{W}_i$  for  $i = 0, \dots, n+1$ . Now the proof is finished.  $\square$

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